

Cosmology for Scalar Fields with $w_\phi < -1$ and Negative Potentials

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ABSTRACT

We study the cosmology of canonically normalized scalar fields that lead to an equation of state parameter of $w_\phi = p_\phi/\rho_\phi < -1$ without violating the weak energy condition: $\rho = \Sigma_i \rho_i \geq 0$ and $\rho_i + p_i \geq 0$. This kind of behavior requires a negative scalar potential V , widely predicted in particle physics. We show that the energy density $\rho_\phi = E_k + V$ takes negative values with an equation of state with $w_\phi < -1$. However, the net effect of the ϕ field on the scale factor is to decelerate it giving a total equation of state parameter $w = p/\rho > w_b = p_b/\rho_b$, where ρ_b stands for any kind of energy density with $-1 \leq w_b \leq 1$, such as radiation, matter, cosmological constant or other scalar field with a potential $V \geq 0$.

The fact that $\rho_\phi < 0$ allows, at least in principle, to have a small cosmological constant or quintessence today as the cancellation of high energy scales such as the electroweak or susy breaking scale. While V is negative $|\rho_\phi|$ is smaller than the sum of all other energy densities regardless of the functional form of the potential V .

We show that the existence of a negative potential leads, inevitable, to a collapsing universe, i.e. to a would be "big crunch". In this picture we would still be living in the expanding universe.

The solution to the FRW equations have a time inversion symmetry around t_0 where $H(t_0) = 0$. This symmetry ensures that at the final time ($t_f = 2t_0$) we will have the same values of all energy densities and the solution to a "big crunch" and "big bang" singularity would be the same.

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1 Introduction

The recent cosmological observations show that the universe is in an accelerating epoch. These observations, together with CMB data, require a dark energy density of $\Omega_{de} \simeq 0.7$ and negative equation of state $w_{de} = p_{de}/\rho_{de} < 0$ [1, 2]. The data suggest that $w_{de} < -2/3$ and the best fit models have w_{de} very close to minus one including the regions around $w_{de} = -1$. Even more, the existing data do not rule out a $w_{de} < -1$ [2]. Causality arguments seem to require $w > -1$ but this is only the case for constant w and in fact it is possible to have $|w| > 1$ without violating causality [6]. The possibility of having $w < -1$ was raised by [4]-[5] where they studied a phantom energy density with $\rho_{ph} > 0$ and $w_{ph} < -1$. From a particle point of view it is unclear how such field can be obtained and it was suggested to have a scalar field with a negative kinetic energy [4]. In this work, we will concentrate on scalar fields with canonical kinetic terms ($E_k \geq 0$) that lead dynamically to regions with $|w| > 1$.

The cosmology of negative potential has become increasingly interesting since many particle physics models and string theory predict anti-de-Sitter "ADS" spaces. The cyclic universe [7] and the ekpyrotic [8] universe models use a negative potential and describe a universe that goes from a contracting phase (leading to a big crunch) to an accelerating universe. The main idea is to give an alternative theory to the big bang singularity. However, it is not yet clear the process of the transition between contracting and accelerating phases.

Scalar fields are widely obtained in particle physics, and they can be either fundamental fields or composite fields (fermion condensates) [10, 12]. They can be used to describe the cosmological constant as quintessence [11] and in all these models the scalar fields have a positive semi-definite potential V [14, 15]. However, negative potential are widely predicted by string theory or by supergravity (sugra) models. The sugra potential is

$$V = e^K \left[F_i (K^{-1})^i_j F^j - 3|W|^2 \right] \quad (1)$$

with $K(\phi, \bar{\phi})$ the Kahler potential, $W(\phi)$ the superpotential and $F_i = K_i W + W_i$, $W_i \equiv dW/d\phi_i$. It is clear that the general vacua of sugra has a negative potential and a lot of model building (in many cases not quite self consistent) has to be done to get a potential $V \geq 0$.

Here, we will study the cosmology of negative potentials and we will show that they lead to an equation of state with $|w_\phi| > 1$ in a natural dynamical way. The stage of $w_\phi < -1$ has a negative energy density $\rho_\phi < 0$ but a positive $\rho_\phi + p_\phi = 2E_k \geq 0$. These models lead inevitably to a collapsing universe and to a would be "big crunch" contrary to phantom fields. Indeed, phantom physics suggest that the final stage of the universe has a divergence of the scale factor at finite time ($a(t) \rightarrow \infty$) [5] called a "big smash" while a "big crunch" ends in a singularity $a(t) \rightarrow 0$, $\rho \rightarrow \infty$.

We do not worry about a big crunch as we do not worry about a big bang since we are working with an effective theory and we would expect that a fundamental theory like strings could have

a graceful exit to the problem of singularities. The behavior of the universe approaching the big crunch singularity is just as bad as the big bang singularity since the universe has, in general, a time inversion symmetry $t_0 + t \rightarrow t_0 - t$ around the point where the Hubble constant is zero $H(t_0) = 0$. So, whatever the solution to the big bang singularity will also solve the big crunch singularity. The evolution of the universe outside the possible singularities may well be described by the effective field theory and general relativity.

An accelerating universe requires a dominating energy density with $w_i < -1/3$, (or more exact $\Omega_i w_i < -1/3$, $\Omega_i = \rho_i / \rho_{Tot}$), and therefore we could naively think that a $w_i < -1$ will increase the acceleration of the universe. However, a canonical normalized scalar field with $w_\phi < -1$ decelerates the universe because in this region the energy density is negative, as we will show later, and its contribution to the acceleration of the scale factor given by

$$\ddot{a} \sim -(\rho_\phi + 3p_\phi) = -\rho_\phi(1 + 3w_\phi) \quad (2)$$

is negative, thus decelerating the expansion of the universe.

Another interesting feature of negative potentials is that it allows, at least in principle, to have a small cosmological constant as the cancellation of much higher energies scales, like the electroweak scale "EW" or susy breaking scale. During all the time that ρ_ϕ is negative we have $|\rho_\phi|$ smaller than the sum of all other energy densities for a long period of time and regardless of the functional form of the potential V .

2 Energy Conditions

The weak energy condition "WEC" states that the total energy density ρ and pressure p obey the inequalities

$$\begin{aligned} \rho + p = \rho(1 + w) &\geq 0 \\ \rho &\geq 0. \end{aligned} \quad (3)$$

where we have used the equation of state $p = w\rho$. If WEC is supplemented with $\rho \geq |p|$ then it is called the dominant energy condition "DEC". The argument for requiring $\rho \geq |p|$ is because for a constant equation of state $w = p/\rho$, w gives the speed of sound and it should not exceed the speed of light $c = 1$, i.e. $|w| \leq 1$. However, if w is not constant then we can have regions with $|w| > 1$ and $|dp/d\rho| > 1$ without violating causality. A full discussion on the sound speed, signal speed and causality for scalar fields is given in [6].

In a flat universe, the second equation in (3) can be understood also from the positivity of the Friedman equation $3H^2 = \rho$ and the first equation is obeyed by matter ($w = 0$), radiation ($w = 1/3$), cosmological constant ($w = -1$) and scalar fields ($\rho + p = 2E_k = \dot{\phi}^2$).

Here we will concentrate on the cosmology behavior of a canonically normalized scalar field ϕ , $E_k = \dot{\phi}^2/2$, with a potential that has a negative minimum $V(\phi)|_{min} < 0$ in a flat universe. Having a scalar field with a negative potential will lead dynamically to a region with $|w_\phi \equiv p_\phi/\rho_\phi| > 1$.

In a flat universe we have

$$H^2 = \frac{1}{3}\rho \quad (4)$$

with the reduced Planck mass $m_p^2 = 1/8\pi G \equiv 1$ and with $\rho = \Sigma_A \rho_A$, where A stands for all kinds of energy densities, matter, radiation, cosmological constant or scalar fields. It is usually stated that a flat universe expands forever and it is infinite but this conclusion is not completely correct. It is based on the assumption that all energy densities are positive $\rho_A > 0$ at finite time. However, if the contribution from some kind of energy has $\rho_A < 0$, then it is possible to have $3H^2 = \rho = 0$ at finite time and the size of the universe will start to decrease afterwards. The behavior is very similar as for a closed universe with positive curvature since it will reach a maximum size and then will start contracting ending in a big crunch.

To see this, let us study the cosmological evolution of ϕ with an arbitrary potential $V(\phi)$. It can be determined from a system of differential equations describing a spatially flat Friedmann–Robertson–Walker universe,

$$\begin{aligned} \dot{H} &= -\frac{1}{2}(\rho + p), \\ \dot{\rho} &= -3H(\rho + p) \end{aligned} \quad (5)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and dot refers to derivative w.r.t. time.

The weak energy condition together with eqs.(5) imply that H is an non increasing quantity, $\dot{H} \leq 0$. Since the scale factor is always non-negative, $a \geq 0$, the maximum size of the universe will be given at $H(t_0) = 0$ with $a(t) \leq a(t_0)$ and afterwards the scale factor will start to shrink until it arrives at a big crunch with $a \rightarrow 0$. For this kind of behavior, it is required that the total energy density vanishes at a finite time. For $H > 0$ the energy density ρ decreases with time and it reaches $\rho(t_0) = 0$ at $H(t_0) = 0$. If this happens at a finite time t_0 then the evolution in eq.(5) leads to a negative H and a increasing ρ for $t > t_0$.

A third possibility is that of a phantom energy. The phantom energy does not comply with the WEC since it has $\rho_{ph} > 0$ but $\rho_{ph} + p_{ph} = \rho_{ph}(1 + w_{ph}) = 2E_k < 0$, i.e. the scalar field has non-canonical (negative) kinetic term. From eqs.(5) we can see that if the phantom field dominates then \dot{H} is positive giving an increasing H and ρ_{ph} . The fact that phantom physics gives a big smash can be understand from the solution of the scale factor valid for w_{ph} constant and when the phantom field starts to dominate and after a period of matter domination. The solution is $a(t) = a(t_m)[-w_{ph} + (1 + w_{ph})(t/t_m)]^{2/3(1+w_{ph})}$ [4]. It is clear that $a \rightarrow \infty$ at the finite time $t = t_m w_{ph}/(1 + w_{ph})$ since the exponent of $a(t)$ is negative ($1 + w_{ph} < 0$) and the term in brackets vanishes. As mentioned above, here we will not work with this kind of energy density since we are only considering standard canonically normalized scalar fields.

It is clear, from the Friedman and WEC equations, that the total energy density must be positive semi-definite but we can have a sector with negative energy densities, say ρ_i , without contradicting the WEC (eq.(3)) as long as the total energy $\rho = \Sigma_A \rho_A \geq 0$. However, even for this type of energy density, which we assume comes from a canonical normalized scalar field we have

$$\rho_\phi + p_\phi = \rho_\phi(1 + w_\phi) = 2E_k \geq 0 \quad (6)$$

with $\rho_\phi = E_k + V, p_\phi = E_k - V$. This inequality is clearly also satisfied for matter, radiation or cosmological constant. Therefore, $\rho_i + p_i \geq 0$ holds for all classes of fluids we are here considering. However, eqs.(6) allows for a negative energy density as long as $w_\phi < -1$, i.e. we can have

$$\rho_\phi < 0 \quad \Leftrightarrow \quad w_\phi < -1 \quad (7)$$

as long as the total $\rho \geq 0$ with the total $w = p/\rho > -1$. We will also have for $\rho_\phi > 0$

$$E_k > -V > 0 \quad \Leftrightarrow \quad w_\phi > 1. \quad (8)$$

The fact that we have $w_\phi < -1$ is not a problem with causality because it is only in the case of w_ϕ constant that w_ϕ can be interpreted as the sound speed of the fluid in which case it must be smaller than the speed of light.

3 Time Inversion Symmetry and Dynamics

Let us now study under which conditions do we have a time inversion symmetry around the point t_0 , where $H(t_0) = 0$. The dynamical equations are

$$H^2(t) = \frac{1}{3} \sum_A \rho_A(t) \quad (9)$$

$$\dot{H}(t) = -\frac{1}{2} \sum_A (\rho_A(t) + p_A(t)), \quad (10)$$

$$\dot{\rho}_A(t) = -3H(t)(\rho_A(t) + p_A(t)) \quad (11)$$

where A stands for all kind of fluids, matter, radiation, cosmological constant or scalar fields.

To have an inversion symmetry it is sufficient to have $\rho_A(t) = \rho_A(t')$ and $p_A(t) = p_A(t')$ (or equivalently ρ_A and $\rho_A(1 + w_A(t)) = \rho_A(t) + p_A(t)$ symmetric) for all fluids for $t = t_0 - \Delta t$ and $t' = t_0 + \Delta t$ (for $t_0 = 0$ we would have a transformation $t \rightarrow t' = -t$). For fluids with constant $-1 \leq w_A \leq 1$, as matter, radiation or cosmological constant, the time symmetry is always satisfied. Therefore, the big crunch is just as problematic as the big bang and the solution could be expected to be the same for both singularities.

If we have scalar fields, then a time inversion symmetry is only possible if $\dot{\phi}^2(t) = \dot{\phi}^2(t')$ for all t , since $\rho_\phi + p_\phi = \rho_\phi(1 + w_\phi) = \dot{\phi}^2$. This will be satisfied for $\dot{\phi}(t) = \pm\dot{\phi}(t')$. If

$$\dot{\phi}(t) = -\dot{\phi}(t') \quad \Leftrightarrow \quad \dot{\phi}(t_0) = 0 \quad \Leftrightarrow \quad \phi(t) = \phi(t') \quad (12)$$

Clearly the time inversion symmetry would be, in this case, valid for all kinds of potentials $V(\phi)$. If $\dot{\phi}(t) = \pm\dot{\phi}(t')$ is satisfied locally around t_0 the dynamics ensures that it is valid for all t . However, if $\dot{\phi}(t_0) \neq 0$, then a symmetric solution is only possible if the potential $V(\phi)$ is symmetric around $\phi(t_0)$. This can be seen from the equation of motion of ϕ

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad (13)$$

with $V' = dV/d\phi$ and at t_0 one has $H(t_0) = 0$ and for $\dot{\phi}(t_0) \neq 0$ a symmetric solution requires

$$\dot{\phi}(t) = \dot{\phi}(t') \quad \Rightarrow \quad \ddot{\phi}(t_0) = 0 \quad \Leftrightarrow \quad V'(t_0) = 0. \quad (14)$$

We conclude that arbitrary initial conditions at t_0 for a scalar field will, in general, break the time inversion symmetry. Only a symmetric initial condition given at t_0 with $\dot{\phi}(t_0) = 0$ will respect the time symmetry independently of the potential V , or for a symmetric potential around $\phi(t_0)$ and arbitrary initial condition for $\dot{\phi}$.

As a consequence of the time inversion symmetry the final time is $t_f \simeq 2t_0 - t_i$, with t_i the initial time which could be set to zero, and it does not make sense to take $t_f \rightarrow \infty$. There will be a big crunch only if there was a big bang singularity. The minimum size of the scale factor is then at $a(t_f) = a(t_i)$, and the energy densities and Hubble parameter will end up at the same value as they started $\rho_A(t_i) = \rho_A(t_f)$, $H(t_i) = -H(t_f)$. So the dynamics for $t > t_0$ is just the same as the dynamics for $t < t_0$. In [9] it was argued that negative potential are as dangerous as unbounded potentials and that the kinetic terms dominates at large times. This is true but the "reasonable" position is to cut off the time at $t_f = 2t_0 - t_i$.

3.1 Solution with constant w

It is useful to study first the evolution of fluids with constant w and later introduce a scalar field with arbitrary potential varying w .

The main features can be obtained from the w_b constant example. From eq.(11) an arbitrary energy density evolves as

$$\rho_b(t) = \rho_{bi} \left(\frac{a(t)}{a_i} \right)^{-3(1+w_b)} \quad (15)$$

with a constant equation of state parameter $-1 \leq w_b \leq 1$ and initial energy density $\rho_{bi} > 0$.

As long as we have an expanding universe $H > 0$, i.e. $a(t) > a_i(t_i)$ for $t > t_i$, and with $1 + w_b > 0$ we have a decreasing amplitude in time of ρ_b with $\rho_b \rightarrow 0$ at late times. This kind of behavior is the usual one for radiation or matter, with $w_b = 1/3, 0$ respectively.

However, as discussed above, we can have $H < 0$. In this case we have a decreasing scale factor $a(t) < a_i(t_i)$ ($t > t_i$) and ρ_b increases with time (decreasing $a(t)$). It is easy to see that we have a symmetric ρ_b under the transformation $t = t_0 - \Delta t \rightarrow t' = t_0 + \Delta t$ with $H(t_0) = 0$ if $a(t) = a(t')$. Eq.(15) can be rewritten as

$$\rho_b(t) = \rho_{b0} \left(\frac{a(t)}{a(t_0)} \right)^{-3(1+w_b)} \quad (16)$$

with $a(t) < a_0(t_0)$ and valid for all t . The energy density ρ_{b0} gives the minimum value of ρ_b .

Let us now include a cosmological constant with negative energy density $\rho_\Lambda = -p_\Lambda = -\Lambda < 0$ and a fluid with constant $|w_b = p_b/\rho_b| < 1$.

In this case we will have a symmetric solution around t_0 .

The equations to be solve are $\dot{\rho}_\Lambda = 0$ and $\rho_b = -3H\rho_b\gamma_b$, with $\gamma_b \equiv 1 + w_b = cte$. Using $H = \pm\sqrt{(\rho_b - \Lambda)/3}$, with the plus sign for $t < t_0$ and the minus sign for $t > t_0$, we can integrate eq.(11) from t to t_0 giving

$$\int \frac{d\rho_b}{\rho_b\sqrt{\rho_b - \Lambda}} = -3\gamma_b |t_0 - t| \quad (17)$$

and

$$\rho_b(t) = \Lambda \tan^2 \left[\frac{\gamma_b\sqrt{3\Lambda}}{2} (t_0 - t) \right] + \Lambda \quad (18)$$

with

$$H(t) = \sqrt{\frac{\Lambda}{3}} \tan \left[\frac{\gamma_b\sqrt{3\Lambda}}{2} (t_0 - t) \right] \quad (19)$$

were we have taken $\rho_b(t_0) = \Lambda$ (i.e. $H(t_0) = 0$). The Hubble parameter $H(t)$ in eq.(19) is positive for $t < t_0$ and it is negative for $t > t_0$. It is easy to see that the solution (18) and (19) are symmetric under time inversion $t_0 - t \rightarrow t_0 + t$ and ρ_b, H blow up at $\frac{\sqrt{3\Lambda}\gamma_b}{2}(t_0 - t) = \pi/2$ since $\tan[\pi/2] = \infty$. The quantities ρ_b, H^2 start at a given initial value and they decrease as long as $t < t_0$ reaching its minimum value at t_0 with $\rho_b(t_0) = \Lambda$ and $H(t_0) = 0$. For $t > t_0$ we have a negative H and ρ_b starts to increase. We would take the final value of t as $t_f - t_0 = t_0 - t_i$ and $\rho_b(t_f) = \rho_b(t_i)$. We can also see that ρ never becomes negative nor ρ_b smaller than Λ .

3.2 Scalar field Dynamics

We will now study the evolution of scalar field with a negative minimum $V|_{min} < 0$ potential with energy density $\rho_\phi = E_k + V$ and pressure $p_\phi = E_k - V$. This situation is completely equivalent as to having a semi-positive definite potential in the presence of a negative energy density.

Considering a scalar field with negative potential but with no other type of energy densities, then WEC requires $E_k + V \geq 0$ and therefore w_ϕ is bounded from below $w_\phi \geq -1$ at all times.

In this case we cannot arrive at the situation where we have $w_\phi < -1$ and $\rho_\phi < 0$, even though we have a negative V but there will be regions with $w_\phi > -1$.

For two (or more) sectors which are only connected via gravity so that $\rho = \rho_b + \rho_\phi$, $p = p_b + p_\phi$ with $\dot{\rho}_A = -3H(\rho_A + p_A)$ and equation of state $p_A = w_A \rho_A$ for $A = b, \phi$. In this case the weak energy principle is $\rho = \rho_b + \rho_\phi \geq 0$ and $\rho_A + p_A = \rho_A(1 + w_A) \geq 0$ for each component separately $A = b, \phi$.

To have an idea on how the evolution of the different energy densities behave in the presence of a scalar field with negative potential we solve eq.(11) for ρ_ϕ with constant w_ϕ and consider the different regions. Of course, w_ϕ is far from being a constant but the generic features are well described by taking w_ϕ constant with different values. We will later allow for a dynamical non constant w_ϕ .

An arbitrary energy density evolves as in eq.(15).

$$\rho_\phi(t) = \rho_{\phi i} \left(\frac{a(t)}{a_i} \right)^{-3(1+w_\phi)}. \quad (20)$$

where $\rho_{\phi i}$, w_ϕ take different values depending in which region they are taken. As long as we have an expanding universe $H > 0$, i.e. $a(t) > a_i$, and we can distinguish two different cases $\rho_{\phi i}$ positive or negative. For $\rho_{\phi i} > 0$ we have $1 + w_\phi > 0$ and a decreasing amplitude of ρ_ϕ . However, as discussed above, we can have $\rho_\phi \leq 0$ with total $\rho \geq 0$. In this case, from eq.(7) we necessarily have $w_\phi < -1$. Still in the region with $H > 0$ we have a decreasing ρ_ϕ but since it is negative $|\rho_\phi|$ grows in time, i.e. it becomes more and more negative. In the mean time the energy density of all other sectors (with $\rho_b > 0$) are also decreasing (in absolute terms), so we will necessarily reach a point when the total energy density vanishes $\rho = \rho_b + \rho_\phi = 0$, $H = 0$.

During all the time that $\rho_\phi < 0$ we have $|\rho_\phi| < \rho_b$, i.e. it is smaller than the sum of all other energy densities for a long period of time and regardless of the functional form of the potential V and the fact that $w_\phi < -1$.

Since H is always decreasing, after $H(t_0) = 0$ we will have a contracting universe with $H < 0$, i.e. $a(t) < a(t_0)$. At this time ρ_ϕ , which is still negative, will start growing with a $w_\phi < -1$ and become less negative. Eventually, ρ_ϕ will become positive and $w_\phi > -1$. The energy density will continue to grow with decreasing scale factor $a(t)$. If we have a symmetric solution for the scalar fields (see eqs.(12) and (14)) than $a(t_f) \rightarrow a_i(t_i)$ and $\rho_\phi \rightarrow \rho_{\phi i}$ where $a_i, \rho_{\phi i}$ are the initial values.

Until now, we have shown the generic evolution of ρ_ϕ considering w_ϕ constant and taking different values depending on the region.

In [9] it was shown that a scalar field with negative potential its kinetic energy will dominate over the scalar potential V but this result depends on the potential. Let us now see under which conditions will the kinetic term dominate the potential. In this approximation we can take in

eq.(13) $V' \ll 3H\dot{\phi}$, i.e.

$$\ddot{\phi} + 3H\dot{\phi} = 0. \quad (21)$$

In order to analytically solve eq.(21) we can take, without loss of generality, the scalar potential $V(\phi) = V_1(\phi) + V_{min}$, with $V_{min} \equiv -\Lambda$ the (constant) value of V at the minimum and $V_1 \geq 0$. In the case that $\rho_b + V_{min} = \rho_b - \Lambda$ dominates over V_1 we can use in eq.(21) H given by eq.(19). Doing so, we have $d\phi/\phi = -3Hdt$ with $\phi \equiv \dot{\phi}$ and integrating the r.h.s. we find

$$\begin{aligned} \int_t^{t_0} \frac{d\phi}{\phi} &= - \int_t^{t_0} 3Hdt \\ \text{Log} \left[\frac{\phi(t_0)}{\phi(t)} \right] &= - \frac{2}{\gamma_b} \text{Cos} \left(\frac{\sqrt{3\Lambda}\gamma_b}{2} (t_0 - t) \right) \end{aligned} \quad (22)$$

giving

$$\dot{\phi}(t) = \dot{\phi}_0 \left[\text{Cos} \left(\frac{\gamma_b \sqrt{3\Lambda}}{2} (t_0 - t) \right) \right]^{\frac{2}{\gamma_b}}. \quad (23)$$

For a matter background $\gamma_b = 1 + w_b = 1$ the exponent in eq.(23) is 2 and we can solve it for $\phi = \int \dot{\phi} dt$ giving

$$\phi(t) = \phi_0 + \frac{t - t_0}{2} + \frac{1}{2\sqrt{3\Lambda}} \text{Sin} \left(\sqrt{3\Lambda} (t_0 - t) \right). \quad (24)$$

We see from eqs.(23) and (24) that the kinetic term and the scalar field (or potential $V = c\phi^n$) have an oscillating behavior and neither of them diverges large at large t . Of course, this result is only approximated since it has been obtained under the hypothesis that the barotropic fluid dominates so that we can use H of eq.(19).

In the other limiting case where $\ddot{\phi} \ll 3H\dot{\phi} = -V'$ with a potential $V = c\phi^n$, we have $d\phi/V' = d\phi/cn\phi^{n-1} = -dt/3H$ and

$$\phi(t_f)^{2-n} - \phi(t_i)^{2-n} = \frac{2n(n-2)c}{3\Lambda} \text{Log} \left[\frac{\text{Sin} \left(\frac{\sqrt{3\Lambda}}{2} (t_0 - t_f) \right)}{\text{Sin} \left(\frac{\sqrt{3\Lambda}}{2} (t_0 - t_i) \right)} \right] \quad (25)$$

for all t (larger or smaller than t_0). In this case there is a divergent behavior at $t \rightarrow t_0$ where $\text{Log}[\text{Sin}[t \rightarrow t_0]] \rightarrow -\infty$ and $|\phi(t)^{2-n}| \rightarrow \infty$. For $n > 2$ one has $\phi(t) \rightarrow 0$ while for $n < 2$ the value is $\phi(t) \rightarrow \infty$.

4 Effective w

For an arbitrary potential V , the quantity $|w_\phi| = (E_k - V)/(E_k + V) < 1$ at all times if V is a non-negative potential since $E_k + V > E_k - V$. However for negative potentials we have regions with $|w_\phi| > 1$, since in this case the terms in the numerator have the same sign, when

V is negative, while in the denominator they have opposite sign. The parameter w_ϕ will then be greater than one for $\rho_\phi = E_k + V > 0$ and smaller than -1 for $\rho_\phi < 0$ as given by eqs.(7) and (8).

As stated in the introduction, the SN1a and CMB data do not ruled out an energy density with $w < -1$. We have seen so far that in order to have $w_\phi < -1$, for a canonical scalar field, we also need to have $\rho_\phi < 0$. So, what is the effect of having a negative energy density with $w_\phi < -1$ in the total equation of state $w = p/\rho$?

Let us consider an energy density given by $\rho = \rho_1 + \rho_2$ with the conditions $\rho > 0, \rho_1 > 0, \rho_2 < 0$ and equations of state $p = w\rho$, $p = p_1 + p_2$ and $p_i = w_i\rho_i$, $i = 1, 2$. As we have seen previously, w_2 must necessarily be smaller than -1 and we take $-1 < w_1 < 1$. The energy density ρ_1 can be either matter, radiation, cosmological constant or a scalar field with non negative potential (e.g. quintessence $-1 \leq w_1 \leq 1$). The equation of state parameter for the total ρ is

$$w = \frac{p}{\rho} = \frac{w_1\rho_1 + w_2\rho_2}{\rho} \quad (26)$$

Since $\rho_1 > \rho = \rho_1 + \rho_2 > 0$ and $w_2\rho_2 > 0$ (both are negative) we have

$$w > w_1\rho_1/\rho > w_1 \quad (27)$$

regardless of the values of ρ_1, ρ_2 . Notice that the value of w_2 is smaller than -1 but the final w is not only greater than -1 but it is greater than w_1 . This means that a negative potential will increase the value w_1 of any energy density component including that of a scalar field with positive energy density or that of a cosmological constant with $\rho_1 = cte$ and $w_1 = -1$.

A scalar field with a negative minimum potential can always be written as $V = V_1 + V_{min}$ with $V_1 \geq 0$ and $V_{min} < 0$ constant. The minimum of the potential is at $V_1 = 0$ and $V = V_{min} < 0$. This separation of the potential for a scalar field is entirely equivalent as to having a scalar field with non negative potential V_1 and kinetic energy $E_k = \dot{\phi}^2/2$ in the presence of a negative cosmological constant given by $\rho_\Lambda = V_{min} = -p_\Lambda < 0$. The equation of state would be

$$|w_1| = \left| \frac{p_1}{\rho_1} \right| = \left| \frac{E_k - V_1}{E_k + V_1} \right| < 1 \quad w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1. \quad (28)$$

The original equation of state $w_\phi = (E_k - V)/(E_k + V) = (E_k - (V_1 + V_{min}))/ (E_k + V_1 + V_{min})$ can be expressed in terms of w_1 and $w_\Lambda = -1$ as

$$w_\phi = w_1 - (1 + w_1) \frac{\rho_\Lambda}{\rho_1 + \rho_\Lambda} \quad (29)$$

and if $\rho_1 + \rho_\Lambda > 0$ we have $w_\phi < -1$ since $\rho_\Lambda/(\rho_1 + \rho_\Lambda) < -1$ (the terms in the denominator have opposite signs) while for $\rho_1 + \rho_\Lambda < 0$ we get $w_\phi > 1$ since $\rho_\Lambda/(\rho_1 + \rho_\Lambda) > 1$ and we recover eqs.(7) and (8).

We conclude that a scalar field with negative potential is equivalent to a positive semi-definite potential in the presence of a negative energy density cosmological constant.

5 Models with Negative Potentials

Let us now analyze the cosmological evolution of scalar fields for different negative potentials.

In the first case, we consider a potential with a simple mass term $V = m^2\phi^2 + V_2$ with $V_2 = -\Lambda < 0$ in the presence of matter. The evolution is shown in fig.1.

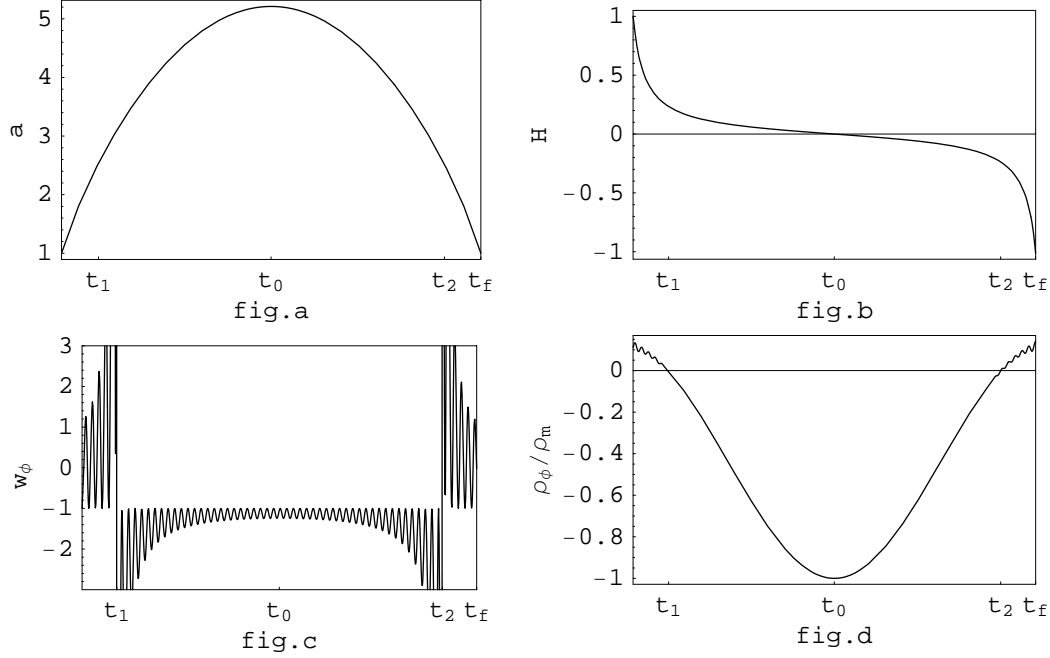


Figure 1: We show the evolution for a scalar field with potential $V = m^2\phi^2 + V_2$, $V_2 = -\Lambda < 0$. The field oscillates around $\phi = 0$ and we have $w_\phi < -1$ for $t_1 < t < t_2$, where t_0 , t_1 and t_2 are defined by $H(t_0) = 0$, $\rho_\phi(t_1) = \rho_\phi(t_2) = 0$. Even though at t_0 , we have $\phi_0 \neq 0$ and $\dot{\phi}_0 \neq 0$, i.e. no symmetric initial conditions, the solution is almost symmetric around t_0 since $|\phi_0| \ll 1$, $|\dot{\phi}_0| \ll 1$.

As given by solving eqs.(9) the field rolls down the potential and oscillates around it. For $t < t_0$ we have an expanding universe with $H > 0$ while for $t > t_0$ we have a contracting universe with $H < 0$. While $\rho_\phi \geq 0$ and $V \geq 0$ we have $-1 \leq w_\phi \leq 1$ and as soon as $V < 0$ we have $w_\phi > 1$. At t_1 and t_2 we have $\rho_\phi(t_1) = \rho_\phi(t_2) = 0$ and we see that for $t_1 < t < t_2$ the equation of state parameter oscillates with amplitude $w_\phi < -1$ and the energy density is $\rho_\phi(t) < 0$. For $t < t_0$ the energy density ρ_ϕ decreases but it increases in magnitude for $t_1 < t < t_0$. We have an almost symmetric phase around t_0 even though $\dot{\phi}(t_0) \neq 0$ this is because the field is oscillating around the minimum with decreasing amplitude for $t \leq t_0$ and the potential itself is symmetric around the $\phi = 0$.

As a second example we consider an inverse power potential with $V = m\phi^{-1} + V_2$, with $V_2 < 0$

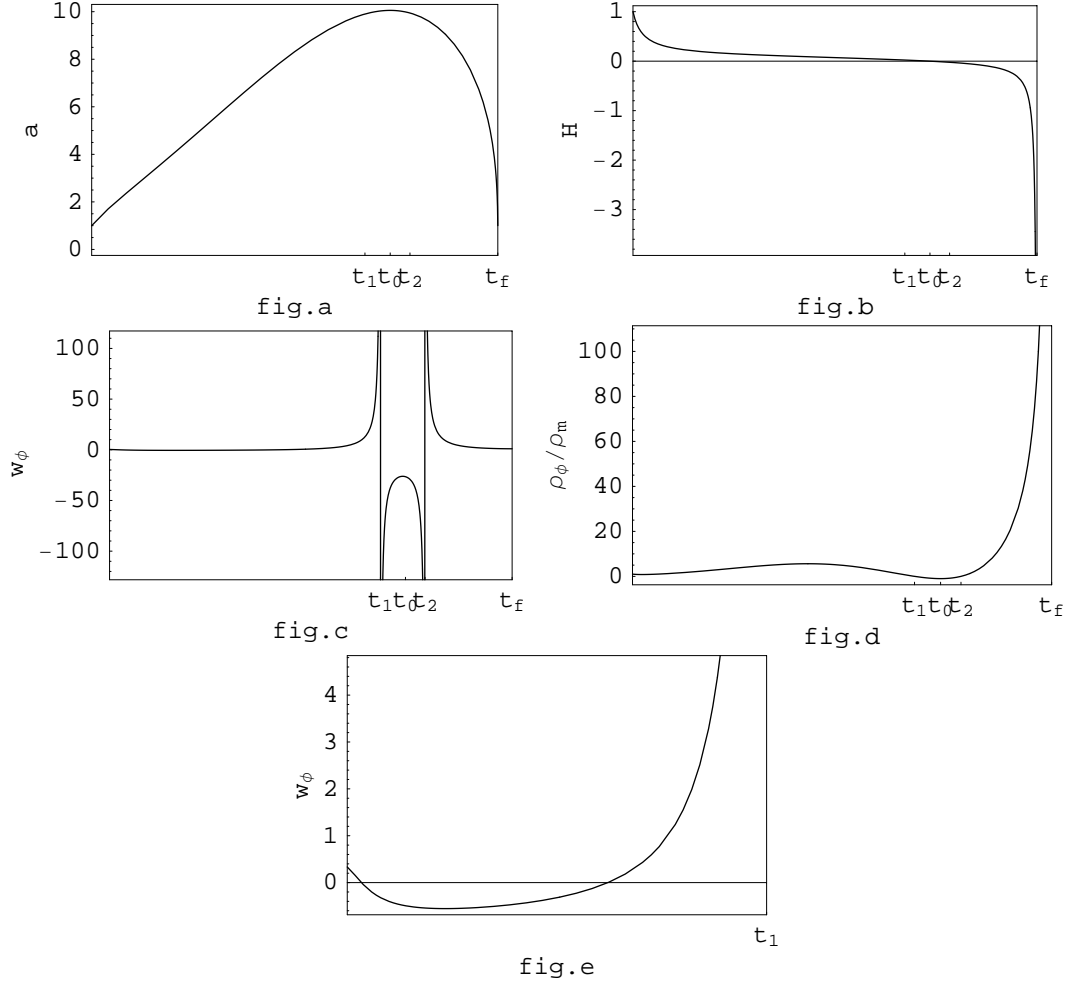


Figure 2: We show the evolution for a scalar potential $V = m\phi^{-1} - \Lambda$ with initial conditions at $\dot{\phi}(t_0) \neq 0$ and the times $t_1 < t_0 < t_2$ defined by $H(t_0) = 0, \rho_\phi(t_1) = \rho_\phi(t_2) = 0$. Notice that the model has no time inversion symmetry around t_0 since $\dot{\phi}(t_0) \neq 0$ and the potential is not symmetric $w_\phi < -1$ for $t_1 < t < t_2$. From fig.e we see that there is a region with $-1 < w_\phi < 0$ where the universe could be in an accelerated epoch.

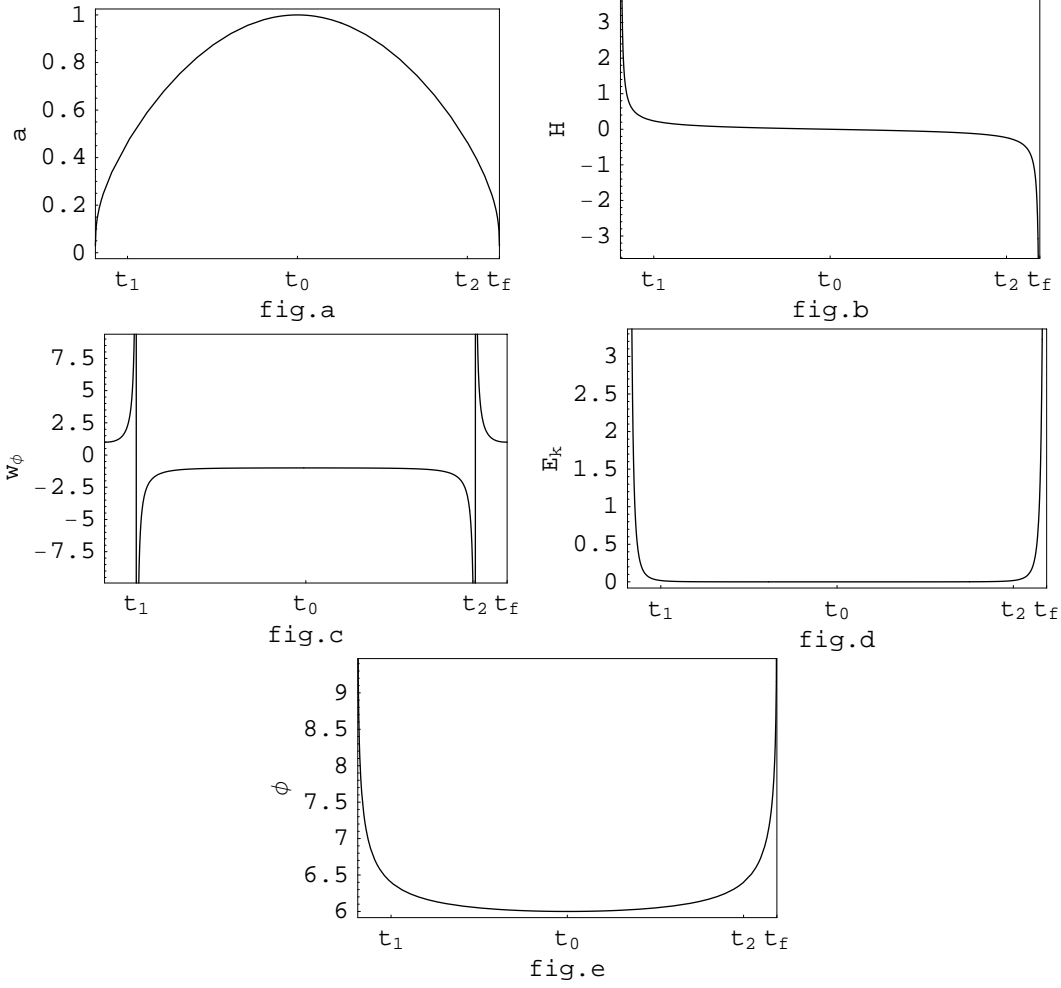


Figure 3: We show the evolution for a scalar potential $V = m\phi^{-1} - \Lambda$ with initial conditions at $\dot{\phi}(t_0) = 0$ and the times $t_1 < t_0 < t_2$ defined by $H(t_0) = 0, \rho_\phi(t_1) = \rho_\phi(t_2) = 0$. Notice that the model is symmetric around t_0 since $\dot{\phi}(t_0) = 0$ even though the potential is not symmetric. Once again, $w_\phi < -1$ for $t_1 < t < t_2$.

constant in the presence of matter with $w = 1$. In this case the potential is clearly not symmetric for any value of ϕ . So, we would expect a symmetric solution only if $\dot{\phi}(t_0) = 0$. In fig.2 and fig.3 we see the evolution of the scale factor a , the Hubble parameter H and the equation of state w_ϕ for different initial conditions. In fig.2 we have $\dot{\phi}(t_0) \neq 0$ and we can see that there is no symmetry for $t < t_0$ and $t > t_0$. However, the generic behavior of the energy densities, ρ_ϕ is still valid, i.e. they decrease for $t < t_0$ and they increase for $t > t_0$. The equation of state parameter $w_\phi < -1$ for $t_1 < t < t_2$.

Notice that for $t < t_1$ the value of w_ϕ is negative, i.e. $-1 < w_\phi < 0$. In the absence of the negative term, we know that inverse power potentials (as $V = m/\phi$) lead to an accelerating epoch and to $\Omega_\phi \rightarrow 1$, dominating the universe at late times and the acceleration of the scale factor does not stop. These kind of potentials are used as quintessence. Here, with the inclusion of a negative term we could have an accelerating scale factor but the period of positive acceleration will necessarily stop.

In fig.3 we show the evolution for the same potential but with $\dot{\phi}(t_0) = 0$. We see that in this case even though the potential is not symmetric around $\phi(t_0)$ we have a symmetric solution since $\phi(t_0 - t) = \phi(t_0 + t)$.

6 Cancellation of Energy Densities

In particle physics the prediction of many scalar fields is quite common. It is certainly more natural to assume the existence of more than one scalar field which will, in general, have different potentials and different cosmological evolutions. So, can quintessence be the sum of two or more different kinds of scalar fields? The answer is clearly yes. There is no theoretical or observational argument against it.

The interesting point is that negative potentials allow, at least in principle, to explain the cosmological scale in terms of a high energy scale (e.g. electroweak or susy breaking scale) since the contribution of a negative potential will cancel that of a positive potential rendering a finite small positive cosmological or quintessence energy [13]. Of course, there is a fine tuning problem such that the difference of two quantities at the high energy scale gives something of the order of today's energy $\Lambda_c \sim 10^{-12} GeV$ and that such a cancellation must take place during nucleosynthesis and today.

However, as we saw above, the evolution of the scalar field with negative potential will necessarily give a region with a small cosmological constant, since for ρ_ϕ negative we have $|\rho_\phi| < \rho_b$ regardless of the functional form of the potential V and the fact that $w_\phi < -1$ which makes $|\dot{\rho}_\phi|$ grow faster than ρ_b . The evolution is such that $\rho_b + \rho_\phi \rightarrow 0$ for any potential V or energy density ρ_b . So it is an attractor solution.

During all the time that $\rho_\phi < 0$ we have $|\rho_\phi| < \rho_b$, i.e. it will get smaller than the sum of

all other energy densities for a long period of time and regardless of the functional form of the potential V .

7 Conclusions

We have seen that a canonically normalized scalar field can lead dynamically to a region with an oscillating equation of state parameter with $w_\phi < -1$. For this to happen a negative potential $V(\phi)$ is needed and the region where $w_\phi < -1 \Leftrightarrow \rho_\phi < 0$ and the WEC is not violated.

During all the time that $\rho_\phi < 0$ we have $|\rho_\phi|$, smaller than the sum of all other energy densities for a long period of time and regardless of the functional form of the potential V . This opens the possibility of having a small cosmological constant today as the result of cancellation of higher energy scales, as EW or susy breaking scales.

The total equation of state parameter w increases its value with the contribution of the scalar with $w_\phi < -1$ and $\rho_\phi < 0$ since it gives a negative contribution to $\ddot{a} \sim -\rho_\phi(1 + 3w_\phi)$.

The solution to the FRW equation have a time inversion symmetry around t_0 where $H(t_0) = 0$. This symmetry ensures that at the final time, $t_f = 2t_0$, we will have the same values of all energy densities and the big crunch singularity will be avoided if there is no big bang singularity.

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